2023-24 MATH2048: Honours Linear Algebra II

Homework 10

Due: 2023-12-04 (Monday) 23:59

For the following homework questions, please give reasons in your solutions. Scan your solutions and submit it via the Blackboard system before due date.

1. Let W be a finite-dimensional subspace of an inner product space V. Show that if T is the orthogonal projection of V on W, then $I - T$ is the orthogonal projection of V on $W^{\perp}$.

2. Let T be a linear operator on a finite-dimensional inner product space V.

   (a) If T is an orthogonal projection, prove that $\|T(x)\|^2 \leq \|x\|^2$ for all $x \in V$. Give an example of a projection for which this inequality does not hold. What can be concluded about a projection for which the inequality is actually an equality for all $x \in V$?

   (b) Suppose that T is a projection such that $\|T(x)\|^2 \leq \|x\|^2$ for $x \in V$. Prove that T is an orthogonal projection.

3. (a) Let A and B be commuting square matrices, i.e., $AB = BA$. Show that the binomial formula can be applied to $(A+B)^n$, i.e.,

   $$(A+B)^n = \sum_{k=0}^{n} \binom{n}{k} A^{n-k} B^k,$$

   where $\binom{n}{k}$ is the binomial coefficient.

   (b) Let A the Jordan block

   $$A = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix},$$

   Find A^4. (Hint: Use part (a).)

4. Let V be the real vector space of functions spanned by the set of real valued functions $\{1, t, t^2, e^t, te^t\}$, and T the linear operator on V defined by $T(f) = f'$.

   Find a basis for each generalized eigenspace of T consisting of a union of disjoint cycles of generalized eigenvectors. Then find a Jordan canonical form J of T.

5. Let $\gamma_1, \gamma_2, ..., \gamma_p$ be cycles of generalized eigenvectors of a linear operator T corresponding to an eigenvalue λ. Prove that if the initial eigenvectors are distinct, then the cycles are disjoint.

The following are extra recommended exercises not included in homework.

1. Let T be a normal operator on a finite-dimensional complex inner product space V. Use the spectral decomposition $\lambda_1 T_1 + \lambda_2 T_2 + ... + \lambda_k T_k$ of T to prove the following results.

   (a) If g is a polynomial, then $g(T) = \sum_{i=1}^{k} g(\lambda_i) T_i$.

   (b) if $T^n = T_0$ for some positive integer n, then $T = T_0$.

   (c) Let U be a linear operator on V. Then U commutes with T if and only if U commutes with each T_i.

   (d) There exists a normal operator U on V such that $U^2 = T$.

   (e) T is invertible if and only if $\lambda_i \neq 0$ for $1 \leq i \leq k$.

   (f) T is a projection if and only if every eigenvalue of T is 1 or 0.

   (g) $T = -T^*$ if and only if every λ_i is an imaginary number.

2. Let T be a normal operator on a finite-dimensional inner product space. Prove that if T is a projection, then T is also an orthogonal projection.

3. Let

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 1 & -1 & 3 \end{pmatrix}$$

   Find a basis for each generalized eigenspace of L_A consisting of a union of disjoint cycles of generalized eigenvectors. Then find a Jordan canonical form J of A.

4. Let T be a linear operator on a vector space V, and let γ be a cycle of generalized eigenvectors that corresponds to the eigenvalue λ. Prove that span(γ) is a T-invariant subspace of V.

5. Let T be a linear operator on a finite-dimensional vector space whose characteristic polynomial splits, and let λ be an eigenvalue of T.

   (a) Suppose that γ is a basis for K_λ consisting of the union of q disjoint cycles of generalized eigenvectors. Prove that $q \leq \dim(E_\lambda)$.

   (b) Let β be a Jordan canonical basis for T, and suppose that $J = [T]_\beta$ has q Jordan blocks with λ in the diagonal positions. Prove that $q \leq \dim(E_\lambda)$.